Notes on the Log-linearization of the Standard RBC model

Igor Francetic, MScE Student, `igor.francetic[at]unil.ch`

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1 Introduction

I wrote these notes during the Spring 2014 exam session preparation at HEC Lausanne (MScE programme). The notes are meant to help through some practical tricks related to the log-linearization of the Standard Real Business Cycle model (RBC), but could also be useful to understand other models involving log-linearization. The model is presented here as taught during the Spring 2014 Business Cycle course (Prof. M. Amand) and it's only one possible shape of the RBC model rather than a general fomulation. The author is solely responsible for all statements made in his work.

2 The standard RBC model (divisible labor)

2.1 General setup

In general,

- Perfect competition in all markets
- The unique good can be consumed or invested
- Price of the unique good normalized to 1
- Capital can be consumed
- No default and no-Ponzi condition in financial market

2.2 Households

2.2.1 Setup

Households are

- rational and forward looking
- infinitely lived
- utility maximizing

The utility function is

$$U_0 = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \right]$$

and has the following properties:

- time separable
- not separable in c and l (otherwise, cross-derivative would be zero)
- l_t is often written as $1 - h_t$

Finally, households can

- Work for a wage w
- Consume or invest the good
- Rent capital to firms for a price d
- Buy and sell bonds b, worth 1 today and $(1 + r)$ tomorrow (contracts enforced)

2.2.2 Problem

Households maximize lifetime utility, as follows

$$\max_{c_t,i_t,b_t,h_t} E_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t,1-h_t)\right]$$

subject to the period budget constraint (b.c.)

$$h_t w_t(1+r_t)b_t+(1+d_t)k_t=c_t+k_{t+1}+b_{t+1}$$

with $i_t=k_{t+1}-k_t$. We can rewrite the b.c. to get

$$c_t=h_t w_t+(1+r_t)b_t+(1+d_t)k_t-k_{t+1}-b_{t+1}$$

and also

$$k_{t+1}=h_t w_t+(1+r_t)b_t+(1+d_t)k_t-c_t-b_{t+1}$$

The problem can be thus written in Bellman form as follows

$$V(k_t,b_t)=\max_{c_t,h_t,b_{t+1}}[u(c_t,1-h_t)+\beta E_0[V(k_{t+1},b_{t+1})]]$$

subject to the same budget constraint (above).
To solve the problem, using the expressions for c_t and k_{t+1}, take FOC's.

FOC_{h_t}

$$u_1(c_t,1-h_t)w_t=u_2(c_t,1-h_t) \tag{1}$$

$FOC_{b_{t+1}}$

$$u_1(c_t,1-h_t)=\beta E_0[V_2(k_{t+1},b_{t+1})] \tag{2}$$

FOC_{c_t}

$$u_1(c_t,1-h_t)=\beta E_0[V_1(k_{t+1},b_{t+1})] \tag{3}$$

Then, using the Envelope Theorem we know that

$$V_1(k_t,b_t)=\beta E_0[V_1(k_{t+1},b_{t+1})](1+d_t) \tag{4}$$

$$V_2(k_t,b_t)=\beta E_0[V_1(k_{t+1},b_{t+1})](1+r_t) \tag{5}$$

Therefore, using (3) in (4) and (2) in (5) we get

$$V_1(k_t,b_t)=u_1(c_t,1-h_t)(1+d_t) \tag{6}$$

$$V_2(k_t,b_t)=u_1(c_t,1-h_t)(1+r_t) \tag{7}$$

(6) and (7) one period ahead can be used in (2) and (3) to obtain the Euler equations

$$u_1(c_t,1-h_t)=\beta E_t[(1+d_{t+1})u_1(c_{t+1},1-h_{t+1})] \tag{8}$$

$$u_1(c_t,1-h_t)=\beta(1+r_{t+1})E_t[u_1(c_{t+1},1-h_{t+1})] \tag{9}$$

while rewriting (1) we get the trade-off represented by the wage

$$w_t=\frac{u_2(c_t,1-h_t)}{u_1(c_t,1-h_t)} \tag{10}$$

(8), (9) and (10) are our optimal time invariant policy equations. Note that d_{t+1} (dividends) are unknown at time t whereas interest r_{t+1} is known.

2.3 Firms

2.3.1 Setup

In the model, firms

- Produce consumption-investment good
- Hire labor at wage w
- Use production function $y_t = A_t f(k_t, h_t)$
- $f(.)$ is CRS, increasing and concave in the arguments
- A_t is the (stochastic and exogenous) level of technology
- Capital depreciates at rate δ
- Riskless environment $\implies$ chose production given actual prices

2.3.2 Problem

Firms are price-taker and maximize profits solving the following problem

$$\max_{k_t,h_t} \pi_t = \max_{k_t,h_t} \left\{ \underbrace{A_t f(k_t, h_t)}_{p_t \times y_t} - w_t h_t - d_t k_t - \delta k_t \right\}$$

Taking the usual FOC's we easily get the solution.

FOC_{k_t}

$$A_t f_1(k_t, h_t) = (d_t + \delta) \tag{11}$$

yielding marginal productivity of capital.

FOC_{h_t}

$$A_t f_2(k_t, h_t) = w_t \tag{12}$$

yielding marginal productivity of labor. (11) and (12) are our policy functions for firms.

2.4 Equilibrium conditions

2.4.1 Definition

An equilibrium in this framework is verified when:

- All Households simultaneously behave optimally
- All Firms simultaneously behave optimally
- All markets clear

2.4.2 Aggregation

In general, with a continuum of agents and firms, consumption and labor aggregation (example) is obtained through

$$C = \int_{i=0}^{1} c_i d_i$$

$$H = \int_{i=0}^{1} h_i d_i$$

In this framework though aggregation is easily obtained because we have identical agents and firms, meaning $C = c$ and $H = h$. Same is true for bonds etc.

2.4.3 Market clearing

In this framework, market clearing conditions are easy to list:

- Financial market clearing: $B = 0$
- Aggregate resource constraint[1]: $Y_t + K_t = C_t + K_{t+1} + \delta K_t$.

3 Solving to the model

The equations we need to solve the model are repeated here for convenience:

Marginal productivity of Capital

$$A_t f_1(k_t, h_t) = (d_t + \delta) \tag{13}$$

Marginal productivity of Labor

$$A_t f_2(k_t, h_t) = w_t \tag{14}$$

Capital market Euler equation

$$u_1(c_t, 1 - h_t) = \beta E_t[(1 + d_{t+1}) u_1(c_{t+1}, 1 - h_{t+1})] \tag{15}$$

Financial markets Euler equation

$$u_1(c_t, 1 - h_t) = \beta(1 + r_{t+1}) E_t[u_1(c_{t+1}, 1 - h_{t+1})] \tag{16}$$

Financial markets clearing condition

$$B = 0 \tag{17}$$

Labor market Euler equation

$$u_1(c_t, 1 - h_t) w_t = u_2(c_t, 1 - h_t) \tag{18}$$

Aggregate resource constraint

$$Y_t + K_t = C_t + K_{t+1} + \delta K_t \tag{19}$$

Of course, we can rewrite (13) and use it in (15)

$$d_t = A_t f_1(k_t, h_t) - \delta$$

At the same time, (14) can be used inside (18).

[1] Here we used the law of motion of capital $K_{t+1} = K_t + I_t$. We could write the aggregate resource constraint as follows: $Y_t = C_t + I_t + \delta K_t$

3.1 Functional forms

In order to develop a proper solution, we need some explicit functional forms for the **Utility function**, the **Production function** and the **technology process**. In particular, we have:

Production function: Cobb-Douglas

$$Y = F(K, H) = A_t K_t^{\alpha} H_t^{1-\alpha}$$

Utility function: Constant Elasticity of Substitution (CES)

$$u(c, h) = \ln c_t + \theta \frac{(1-h_t)^{1-\gamma} - 1}{(1-\gamma)}$$

Technology stochastic process: AR(1)

$$\ln A_{t+1} = \rho \ln A_t + \epsilon \text{ with } \epsilon \sim N(0, \sigma^2)$$

Therefore we can now rewrite (13)-(19) with explicit functional forms, as follows:

$$d_t = A_t \alpha K_t^{\alpha-1} H_t^{1-\alpha} - \delta \tag{20}$$

$$w_t = A_t(1-\alpha) K_t^{\alpha} H_t^{-\alpha} \tag{21}$$

$$\frac{1}{C_t} = \beta E_t \left[(1 + A_{t+1} \alpha K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} - \delta) \frac{1}{C_{t+1}} \right] \tag{22}$$

$$\frac{1}{C_t} = \beta(1 + r_{t+1}) E_t \left[\frac{1}{C_{t+1}} \right] \tag{23}$$

$$B = 0 \tag{24}$$

$$A_t(1-\alpha) K_t^{\alpha} H_t^{-\alpha} = \theta C_t (1 - H_t)^{-\gamma} \tag{25}$$

$$Y_t + K_t = C_t + K_{t+1} + \delta K_t \tag{26}$$

3.2 Methodology

The method used here to simplify the system of equations is log-linearization, which is a first-order approximation around the steady-state. Precisely because of that, it's valid only in the neighborhood of the steady-state. Moreover, since it's a linear instead of quadratic approximation, the risk dimension is totally flattened. The general procedure can be summarized by the following algorithm:

1. Define steady-state variables: x^*
2. Define "hat" variables as log deviations from steady-state: $\hat{x} = \ln x - \ln x^*$
3. Rewrite every variable x in the following form: $x \to x^* e^{\hat{x}}$
4. Solve, using one of the following whenever necessary:
   - Take log
   - For x small, $\log(1 + x) \approx x$
   - The same is true for $e^x \approx (1 + x)$
   - Simplify using the steady-state relations

3.2.1 Log-linearizing the production function

The steady-state relation is just the production function (starred).
Thus, we start rewriting it as mentioned above:

$$Y^* e^{\hat{Y}_t} = A^* e^{\hat{A}_t} (K^* e^{\hat{K}_t})^\alpha (H^* e^{\hat{H}_t})^{1-\alpha}$$

The steady-state relation cancels out, thus we're left with

$$e^{\hat{Y}_t} = e^{\hat{A}_t} (e^{\hat{K}_t})^\alpha (e^{\hat{H}_t})^{1-\alpha}$$

Now, taking logs we get the desired result

$$\boxed{\hat{Y}_t = \hat{A}_t + \alpha \hat{K}_t + (1-\alpha)\hat{H}_t} \tag{27}$$

3.2.2 Log-linearizing the financial market Euler equation

We start with equation (23). In the steady state, $C_t = C_{t+1} = C^*$, hence

$$\beta = \frac{1}{(1+r^*)}$$

Rewriting (22) in the usual solution way (above) we get

$$\frac{1}{C^* e^{\hat{C}_t}} = \beta(1 + r^* e^{\hat{r}_t}) E_t \left[\frac{1}{C^* e^{\hat{C}_{t+1}}} \right]$$

Manipulating a bit we get

$$\frac{1}{(1 + r^* e^{\hat{r}_t})} = \beta e^{\hat{C}_t} E_t \left[\frac{1}{e^{\hat{C}_{t+1}}} \right]$$

Now, applying the trick $e^x \approx (1+x)$ we get

$$\frac{1}{\beta[(1+r^*) + r^* \hat{r}_t]} = E_t \left[\frac{e^{\hat{C}_t}}{e^{\hat{C}_{t+1}}} \right]$$

Solving further (in particular applying the steady-state relation for the discount factor) we get

$$\frac{1}{1 + \beta r^* \hat{r}_t} = E_t \left[e^{\hat{C}_t - \hat{C}_{t+1}} \right]$$

Using $e^x \approx (1+x)$ and taking $e^{\hat{C}_t}$ on LHS we have

$$\frac{1}{e^{\beta r^* \hat{r}_t + \hat{C}_t}} = E_t \left[e^{-\hat{C}_{t+1}} \right]$$

$$e^{\beta r^* \hat{r}_t + \hat{C}_t} = E_t \left[e^{\hat{C}_{t+1}} \right]$$

Now, taking logs on both sides we finally get the desired result

$$\boxed{\beta r^* \hat{r}_t + \hat{C}_t = E_t \left[\hat{C}_{t+1} \right]} \tag{28}$$

3.2.3 Log-linearizing the capital market Euler equation

We start with equation (22), with d_{t+1} instead of the ugly functional form expression. Thus

$$\frac{1}{C_t} = \beta E_t \left[\frac{(1 + d_{t+1})}{C_{t+1}} \right]$$

Afterwards, the procedure is exactly the same as the one described to obtain (28). The results are also similar, with the only difference that here we've an expectation because dividends are uncertain. Hence

$$\boxed{\beta d^* E_t[\hat{d}_t] + \hat{C}_t = E_t \left[\hat{C}_{t+1} \right]} \tag{29}$$

3.2.4 Log-linearizing the labor market Euler equation

We start from equation (25). The steady-state expression is simply

$$A^*(1-\alpha)K^{*(\alpha)}H^{*(-\alpha)} = \theta C^*(1-H^*)^{-\gamma} \tag{30}$$

Rewriting (25) in star-hat form yields

$$A^* e^{\hat{A}_t}(1-\alpha)(K^* e^{\hat{K}_t})^{\alpha}(H^* e^{\hat{H}_t})^{-\alpha} = \theta C^* e^{\hat{C}_t}(1 - H^* e^{\hat{H}_t})^{-\gamma}$$

On the LHS we need a $(1 - H^*)^{-\gamma}$ to have the steady-state. Thus we can multiply and divide, simplifying as follows

$$A^* e^{\hat{A}_t}(1-\alpha)(K^* e^{\hat{K}_t})^{\alpha}(H^* e^{\hat{H}_t})^{-\alpha} = \theta C^*(1-H^*)^{-\gamma} e^{\hat{C}_t}(1-H^* e^{\hat{H}_t})^{-\gamma} \frac{1}{(1-H^*)^{-\gamma}}$$

$$e^{\hat{A}_t}(e^{\hat{K}_t})^{\alpha}(e^{\hat{H}_t})^{-\alpha} = \frac{e^{\hat{C}_t}(1 - H^* e^{\hat{H}_t})^{-\gamma}}{(1-H^*)^{-\gamma}}$$

$$e^{\hat{A}_t + \alpha\hat{K}_t - \alpha\hat{H}_t - \hat{C}_t} = \frac{(1 - H^* e^{\hat{H}_t})^{-\gamma}}{(1-H^*)^{-\gamma}}$$

Now, taking logs

$$\hat{A}_t + \alpha\hat{K}_t - \alpha\hat{H}_t - \hat{C}_t = -\gamma \log \left[\frac{(1 - H^* e^{\hat{H}_t})}{(1-H^*)} \right]$$

Applying the usual $e^x \approx (1+x)$ we get

$$\hat{A}_t + \alpha\hat{K}_t - \alpha\hat{H}_t - \hat{C}_t = -\gamma \log \left[\frac{(1 - H^*(1 + \hat{H}_t))}{(1-H^*)} \right]$$

$$\hat{A}_t + \alpha\hat{K}_t - \alpha\hat{H}_t - \hat{C}_t = -\gamma \log \left[\frac{(1 - H^*(1 + \hat{H}_t))}{(1-H^*)} \right]$$

$$\hat{A}_t + \alpha\hat{K}_t - \alpha\hat{H}_t - \hat{C}_t = -\gamma \log \left[\frac{(1 - H^* + H^*\hat{H}_t)}{(1-H^*)} \right]$$

$$\hat{A}_t + \alpha\hat{K}_t - \alpha\hat{H}_t - \hat{C}_t = -\gamma \log\left[1 + \frac{H^*\hat{H}_t}{(1-H^*)}\right]$$

Apply $\log(1+x) \approx x$ to get

$$\hat{A}_t + \alpha\hat{K}_t - \alpha\hat{H}_t - \hat{C}_t = -\gamma\frac{H^*\hat{H}_t}{(1-H^*)}$$

Further note that $\hat{A}_t + \alpha\hat{K}_t - \alpha\hat{H}_t = \hat{Y}_t - \hat{H}_t$.Thus

$$\hat{Y}_t - \hat{H}_t - \hat{C}_t = -\gamma\frac{H^*\hat{H}_t}{(1-H^*)}$$

Solving further

$$\hat{Y}_t - \hat{C}_t = \hat{H}_t\left(1 - \gamma\frac{H^*}{(1-H^*)}\right)$$
$$\hat{Y}_t - \hat{C}_t = \hat{H}_t\left(\frac{1 - H^* - \gamma H^*}{(1-H^*)}\right)$$
$$\hat{Y}_t - \hat{C}_t = \hat{H}_t\left(\frac{1 - H^*(1-\gamma)}{(1-H^*)}\right)$$

Finally, we have the desired result:

$$\boxed{\hat{Y}_t - \hat{H}_t\frac{1 + H^*(\gamma - 1)}{1 - H^*} - \hat{C}_t = 0} \tag{31}$$

3.2.5 Log-linearizing the aggregate resource constraint

We start with equation (26), that is

$$Y_t + K_t = C_t + K_{t+1} + \delta K_t$$

Steady-state is defined as

$$Y^* + K^* = C^* + K^* + \delta K^*$$
$$Y^* = C^* + \delta K^*$$

Usual "star-hat" transformation yields

$$Y^*e^{\hat{Y}_t} + K^*e^{\hat{K}_t} = C^*e^{\hat{C}_t} + K^*e^{\hat{K}_{t+1}} + \delta K^*e^{\hat{K}_t}$$

Rearranging terms we get

$$K^*e^{\hat{K}_{t+1}} = Y^*e^{\hat{Y}_t} + K^*e^{\hat{K}_t} - C^*e^{\hat{C}_t} - \delta K^*e\hat{K}_t$$

Applying $e^x \approx (1+x)$ we get

$$K^*(1 + \hat{K}_{t+1}) = Y^*(1 + \hat{Y}_t) + K^*(1 + \hat{K}_t) - C^*(1 + \hat{C}_t) - \delta K^*(1 + \hat{K}_t)$$

Simplifying (redundant terms and steady-state relation) we get

$$K^*\hat{K}_{t+1} = Y^*\hat{Y}_t + K^*\hat{K}_t - C^*\hat{C}_t - \delta K^*\hat{K}_t$$

Rearranging we get the desired result

$$\boxed{\hat{K}_{t+1} = \frac{Y^*}{K^*}\hat{Y}_t - \frac{C^*}{K^*}\hat{C}_t + (1-\delta)\hat{K}_t} \tag{32}$$

3.2.6 Log-linearizing the technology process

We start with the technology process equation, that is

$$\ln A_{t+1} = \rho \ln A_t + \epsilon$$

The steady-state is defined as

$$\ln A^* = \rho \ln A^*$$

Written in "star-hat" form is

$$\ln A^* e^{\hat{A}_{t+1}} = \rho \ln A^* e^{\hat{A}_t} + \epsilon$$

Solving

$$\ln A^* + \hat{A}_{t+1} = \rho \ln A^* + \rho \hat{A}_t + \epsilon$$

Simplifying the steady-state we finally get

$$\boxed{\hat{A}_{t+1} = \rho \hat{A}_t + \epsilon} \tag{33}$$

3.3 Set of log-linearized equations

To sum up, the standard RBC model (divisible labor) with assumptions in section 2.5.1 boils down to the set of log-linearized equations derived above, repeated here for convenience:

$$\hat{Y}_t = \hat{A}_t + \alpha \hat{K}_t + (1-\alpha)\hat{H}_t \tag{34}$$

$$\beta r^* \hat{r}_t + \hat{C}_t = E_t \left[\hat{C}_{t+1} \right] \tag{35}$$

$$\beta d^* E_t[\hat{d}_t] + \hat{C}_t = E_t \left[\hat{C}_{t+1} \right] \tag{36}$$

$$\hat{Y}_t - \hat{H}_t \frac{1 + H^*(\gamma - 1)}{1 - H^*} - \hat{C}_t = 0 \tag{37}$$

$$\hat{K}_{t+1} = \frac{Y^*}{K^*}\hat{Y}_t - \frac{C^*}{K^*}\hat{C}_t + (1-\delta)\hat{K}_t \tag{38}$$

$$\hat{A}_{t+1} = \rho \hat{A}_t + \epsilon \tag{39}$$

Equations (34)-(39) can be solved easily with standard methods used to solve a system of linear equations.