Resumé on DF and ADF Unit Root Tests

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1 Introduction

This brief resume has been written during the Applied Time Series Analysis exam (Fall 2012) preaparation, during the MScE programme in HEC Lausanne. The aim is to sum up in a concise way the main results needed to understand and use DF and ADF unit root tests. The derivation of the formulas are not part of this paper, wich is based mainly on J. D. Hamilton's book[1] and inspired by the slides and the lecture notes provided by Prof. F. Pelgrin for the course. Section 2 will treat the simple Dickey and Fuller test, while section 3 will discuss its parametric correction[2], the ADF test. For each section, first subsection provides general informations, second illustrates the test regression needed, third discusses test hypotheses, fourth focuses on the test statistics and critical values while the last offers a complete step by step testing procedure. The author is solely responsible for all statements made in his work.

2 Dickey and Fuller test

2.1 General

The Dickey and Fuller test[3] (DF) is the simpliest test used to check the presence of unit root in the AR lag polinomial. The DF is valid only in an AR(1) framework, wich is quite unusual in economic and finance (time series). If the true d.g.p.[4] is an AR of higher order, the DF test will fail due to omitted variable bias wich makes the errors depart from the wwn[5] assumption needed for the DF test regression. The DF test works with a null hypothesis of nonstationarity and is part of the so-called "first generation unit root tests".

[1] Hamilton J. D. (1994), "Time Series Analysis", Princeton University Press
[2] Needed to account for higher order autocorrelation
[3] Dickey, D.A. and W.A. Fuller (1979), "Distribution of the Estimators for Autoregressive Time Series with a Unit Root," Journal of the American Statistical Association, 74, p. 427–431
[4] Meaning data generating process
[5] Meaning weak white noise

2.2 Test regression(s)

The DF test is based on the coefficients of an OLS[6] regression. There are three main regression settings. Let X_t be the time series of interest; the three models are defined as follows:

M1: $X_t = \rho X_{t-1} + \epsilon_t$ or[7] $\Delta X_t = \phi X_{t-1} + \epsilon_t$
This model is a simple AR(1) without drift and without time trend.

M2: $X_t = \rho X_{t-1} + \alpha + \epsilon_t$ or $\Delta X_t = \phi X_{t-1} + \alpha + \epsilon_t$
Here a costant drift (α) is added with respect to M1.

M3: $X_t = \rho X_{t-1} + \alpha + \beta t + \epsilon_t$ or $\Delta X_t = \phi X_{t-1} + \alpha + \beta t + \epsilon_t$
With respect to M2, a time trend (βt) is added.

The two representations (using ρ or ϕ are eqivalent). The choice between the model(s) is not obvious at the beginning. In subsection 2.5 a testing procedure will be proposed.

2.3 DF test hypotheses

As mentioned above in section 2.1, DF test is based on a null hypothesis of nonstationarity. This means that under H_0, the parameter ρ will equal 1, respectively $\phi = 0$.

Moreover, for M2 and M3 we need joint tests and hypotheses as well, since we have more than one estimated parameter. Those are provided below as well, will the complete procedure is exposed in sections 2.5. Starting from our 3 models, we identify four different cases of interest:

1. **Uses M1**
   $H_0 : \rho = 1 \Longleftrightarrow \phi = 0$
   $H_A : \rho < 1 \Longleftrightarrow \phi = 0$
2. **Uses M2**
   $H_0 : \rho = 1,\, \alpha = 0 \Longleftrightarrow \phi = 0,\, c = 0$
   $H_A : \rho < 1,\, \alpha = 0 \Longleftrightarrow \phi < 0,\, c = 0$
3. **Uses M2**
   $H_0 : \rho = 1,\, \alpha \neq 0 \Longleftrightarrow \phi = 0,\, \alpha \neq 0$
   $H_A : \rho < 1,\, \alpha \neq 0 \Longleftrightarrow \phi < 0,\, \alpha \neq 0$
4. **Uses M3**
   $H_0 : \rho = 1,\, \alpha = 0\, \beta = 0 \Longleftrightarrow H_0 : \phi = 0,\, \alpha = 0\, \beta = 0$
   $H_A : \rho < 1,\, \alpha = 0\, \beta = 0 \Longleftrightarrow H_0 : \phi < 0,\, \alpha = 0\, \beta = 0$
   Here, remaining in the M3 case, we might split the situation in two separate cases:
   (a) We consider explicitly $\alpha = 0$, testing the absence of drift. Thus: $H_0 : \rho = 1,\, \alpha = 0\, \beta = 0 \Longleftrightarrow H_0 : \phi = 0,\, \alpha = 0\, \beta = 0$
   (b) We leave open the hypothesis on α[8], testing the hypothesis with or without drift: $H_0 : \rho = 1,\, \beta = 0 \Longleftrightarrow H_0 : \phi = 0,\, \beta = 0$

2.4 DF test statistics and critical values

The two basic test statistics are:

- **Student test statistic**: $t_{\rho=1} = \frac{\hat{\rho}-1}{se(\hat{\rho})}$ or $t_{\phi=0} = \frac{\hat{\phi}}{se(\hat{\phi})}$
  Note that that Student test in DF framework is not based on the ordinary statistical tables (Normal). Critical values were tabulated by Dickey and Fuller for selected period lenghts and confidence levels[9]. Still, in **case 3** one can use the normal t tables because the asymptotical distribution of the test is a standard Normal.

[6] Ordinary least squares
[7] the second representation is obtained simply by subtracting X_{t-1} from the former, as follows:
Step#0: $X_t = \rho X_{t-1} + \epsilon_t$ (the first representation)
Step#1: $X_t - X_{t-1} = \rho X_{t-1} - X_{t-1} + \epsilon_t$
Step#2: $\Delta X_t = (\rho - 1) X_{t-1} + \epsilon_t$ where $\Delta X_t = X_t - X_{t-1}$ and finally
Step#3: $\Delta X_t = \phi X_{t-1} + \epsilon_t$ where $\phi = \rho - 1$
The same logic applies to M2 and M3.
[8] We can write it $\alpha = \alpha$, but we simply do not control for it
[9] See Hamilton (1994), Appendix B, table B5, page 762. Some tables can be found here: http://www.iei.liu.se/nek/ekonometrisk-teori-7-5-hp-730a07/labbar/1.233753/dfdistab7b.pdf

- **Normalized bias statistic**: $N_{\rho=1} = T * (\hat{\rho} - 1)$ or $N_{\phi=0} = T * \hat{\phi}$
  The critical values, different from the previous ones, are provided by Hamilton[10].

Both tests are left one-sided[11], thus:
. H_0 is rejected if $t < t^*_\alpha$ or $N < N_\alpha*$
. H_0 is not rejected if $t > t^*_\alpha$ or $N > N^*_\alpha$

Moreover, for the testing procedure described in section 2.5, some more tests are needed. One can conduct them as follows.

- **Joint test (F):**
  The F test statistic is given by $F_{(c,T-k)} = \frac{SSR^R - SSR^U}{SSR^R} * \frac{T-k}{c}$
  where $k =$ number of parameters, $c =$ number of costraints and $T =$ is the period lenght
  We distinguish between 3 cases:

  - **Case 2**: likelihood test to test jointly the following hypotheses:
    $H_0 : \rho = 1,\, \alpha = 0 \iff \phi = 0,\, \alpha = 0$ the Restricted model is: $X^R_t = X_{t-1} + \epsilon_t$ or $\Delta X^R_t = X_{t-1} + \epsilon_t$
    $H_A : \rho \neq 1,\, \alpha \neq 0 \iff \phi \neq 0,\, \alpha \neq 0$ the Unrestricted model is: $X^U_t = \rho X_{t-1} + \alpha + \epsilon_t$ or $\Delta X^U_t = \rho X_{t-1} + \alpha + \epsilon_t$
    Again, the critical values are provided by Hamilton[12] and are different from the usual Fisher tabs.
  - **Case 4a**: likelihood test to test jointly the following hypotheses:
    $H_0 : \rho = 1,\, \beta = \alpha = 0 \iff \phi = \alpha = \beta = 0$ the Restricted model is: $X^R_t = X_{t-1} + \epsilon_t$ or $\Delta X^R_t = X_{t-1} + \epsilon_t$
    $H_A : \rho \neq 1,\, \beta \neq \alpha \neq 0 \iff \phi \neq \alpha \neq \beta \neq 0$ the Unrestricted model is $X^U_t = \rho X_{t-1} + \alpha + \beta t + \epsilon_t$ or $\Delta X^U_t = \phi X_{t-1} + \alpha + \beta t + \epsilon_t$
    Critical values are provided by Dickey and Fuller[13] and are different from the usual Fisher tabs.
  - **Case 4b**: likelihood test to test jointly the following hypotheses:
    $H_0 : \rho = 1,\, \beta = 0 \iff \phi = \beta = 0$ the Restricted model is: $X^R_t = X_{t-1} + \epsilon_t$ or $\Delta X^R_t = X_{t-1} + \epsilon_t$
    $H_0 : \rho \neq 1,\, \beta \neq 0 \iff \phi \neq \beta \neq 0$ the Restricted model is: $X^U_t = \rho X_{t-1} + \beta t + \epsilon_t$ or $\Delta X^R_t = \phi X_{t-1} + \alpha + \beta t + \epsilon_t$
    Critical values are provided by Dickey and Fuller[14] and are different from the usual Fisher tabs.

Obiously, to conduct the tests one needs SSR^R and SSR^U. The F tests are right one-sided, thus the interpretation is:
. H_0 is rejected if $F > F^*_\alpha$
. H_0 is not rejected if $F < F^*_\alpha$

- **Significance test (Student)**
  Here's about testing separately the significance of β and α. Again, the sense of those tests will become clear in section 2.5.

  - **Case 2, significance of** α ($H_{0:}\alpha = 0$):
    One conducts a simple Student test and compares the symmetrical test with the critical values provided by Dickey and Fuller [15], wich is different from.
  - **Case 4a, significance of** α ($H_0 : \alpha = 0$):
    One conducts a simple Student test and compares the symmetrical test with the critical values provided by Dickey and Fuller [16], wich is different from **Case 2**.
  - **Case 4a, significance of** β ($H_0 : \beta = 0$):
    One conducts a simple Student test and compares the symmetrical test with the critical values provided by Dickey and Fuller [17], wich is different from **Case 2** and **Case 4a for** α.

Now, we're ready to illustrate the whole testing procedure for DF in all the different steps in the next section.

[10] See Hamilton (1994), Appendix B, table B6, page 763. Some tables can be found here: http://www.iei.liu.se/nek/ekonometrisk-teori-7-5-hp-730a07/labbar/1.233753/dfdistab7b.pdf
[11] Here α stands for the confidence level and not for the constant term
[12] See Hamilton (1994), Appendix B, table B7, case 2, page 764. Some tables can be found here: http://www.iei.liu.se/nek/ekonometrisk-teori-7-5-hp-730a07/labbar/1.233753/dfdistab7b.pdf
[13] The critical values are tabulated in Dickey and Fuller (1979), p. 1063, Table V.
[14] The critical values are tabulated in Dickey and Fuller (1979), p. 1063, Table VI.
[15] The critical values are tabulated in Dickey and Fuller (1979), p. 1062, Table I.
[16] The critical values are tabulated in Dickey and Fuller (1979), p. 1062, Table II.
[17] The critical values are tabulated in Dickey and Fuller (1979), p. 1062, Table III.

2.5 Complete testing procedure

First of all, working with ϕ is more convenient for testing.

Step#1 Start with model 3. Test for the Unit root ($H_0 : \rho = 1$ or $\phi = 0$)

Step#1.1 If you do not reject the null: test jointly the three coefficients ($H_0 : \rho = 1,\ \beta = \alpha = 0 \Longleftrightarrow \phi = \alpha = \beta = 0$)
If you reject the null, it's trend stationary. If you don't reject the null, move to model 2 (model is not correctly specified)

Step#1.2 If you reject the null: test the significance of β alone ($H_0 : \beta = 0$)
If you reject the null, it's trend stationary. If you don't reject the null, move to model 2 (model misspecificated)

Step#1.3 If from 1.1 or 1.2 results a trend stationary series, analyze ACF and PACF of the residuals
If doesn't look like a wwn, there is still correlation in the resisuals and probably ADF is necessary

Step#2 Back to model 2. Test for Unit root ($H_0 : \rho = 1$ or $\phi = 0$)

Step#2.1 If you do not reject the null: test jointly the two coefficients ($H_0 : \rho = 1,\ \alpha = 0 \Longleftrightarrow \phi = \alpha = 0$)
If you reject the null, move to model 1 (model is not correctly specified). If you don't reject the null, you have a unit root. Process is I(1)

Step#2.2 If you reject the null: test the significance of α alone ($H_0 : \alpha = 0$)
If you reject the null, it's weakly stationary and I(0). If you don't reject the null, move to model 2 (model misspecificated)

Step#3.2 If from 2.1 or 2.2 results a weakly stationary series, analyze ACF and PACF of the residuals
If doesn't look like a wwn, there is still correlation in the resisuals and probably ADF is necessary

Step#4 Back to model 1. Test for Unit root ($H_0 : \rho = 1$ or $\phi = 0$)

Step#4.1 If you do not reject the null: you have a unit root. Process is I(1)

Step#4.2 If you reject the null: check residuals properties (ACF and PACF)

3 Augmented Dickey and Fuller test

3.1 General

The ADF is a parametric correction of DF test, wich simply takes into account more autoregressive lags. In fact, DF is valid for AR(1) processes while in most economic and financial time series you have more complicated patterns. The result is that, after a DF testing procedure wich characterizes our series, an analysis of the residuals reveals serial correlation. ADF tries to correct this issue. Finally, note that pretty much all the intuitions of DF are valid for ADF.

3.2 Test regression(s)

Obviously, test regressions in ADF are different from those in DF since we need to take into account more AR coefficients. Thus, starting from the simple representation of the series $X_t = \sum_{j=1}^{p} \rho_j X_{t-j} - \epsilon_t$ and applying the Beveridge-Nelson decomposition we get $X_t = (1 - \phi(1))X_{t-1} + \sum_{j=1}^{p-1} \varphi_j \Delta X_{t-j} + \epsilon_t$. This is the model corresponding to M1 in the DF framework; from now on one could apply the same logic of DF and obtain:

M1: $X_t = (1 - \phi(1))X_{t-1} + \sum_{j=1}^{p-1} \alpha_j \Delta X_{t-j} + \epsilon_t$ or[18] $\Delta X_t = \phi^*(1)X_{t-1} + \sum_{j=1}^{p-1} \varphi_j \Delta X_{t-j} + \epsilon_t$

M2: $X_t = (1 - \phi(1))X_{t-1} + \sum_{j=1}^{p-1} \alpha_j \Delta X_{t-j} + \alpha + \epsilon_t$ or $\Delta X_t = \phi^*(1)X_{t-1} + \sum_{j=1}^{p-1} \varphi_j \Delta X_{t-j} + \alpha + \epsilon_t$

M3: $X_t = (1 - \phi(1))X_{t-1} + \sum_{j=1}^{p-1} \alpha_j \Delta X_{t-j} + \alpha + \beta t + \epsilon_t$ or $\Delta X_t = \phi^*(1)X_{t-1} + \sum_{j=1}^{p-1} \varphi_j \Delta X_{t-j} + \alpha + \beta t + \epsilon_t$

Where $\phi = (1 - \phi_1 - \phi_2 - ... - \phi_p)$ and $\phi^* = -\phi$. All models are estimated again with OLS.

[18] Again, subtracting X_{t-1} from both sides.

3.3 ADF test hypotheses

We'll work only with the second representation, wich involves $\phi^*(1)$ instead of $1 - \phi(1)$. The four cases of interest for the test hypotheses are identical to DF, but the parameter of interest is now $\phi^*(1)$ instead of ρ or ϕ. Here I'll write only null hypotheses.

1. **Uses M1**
   $H_0 : \phi^*(1) = 0$
2. **Uses M2**
   $H_0 : \phi^*(1) = 0,\ \alpha = 0$
3. **Uses M2**
   $H_0 : \phi^*(1) = 0,\ \alpha \neq 0$
4. **Uses M3**
   (a) $H_0 : \phi^*(1) = 0,\ \alpha = 0,\ \beta = 0$
   (b) $H_0 : \phi^*(1) = 0,\ \beta = 0$

3.4 ADF test statistics and critical values

Apart from the test statistic on $\phi^*(1)$, surprisingly it'all the same as DF[19].

- **Student test statistic**: $t_{\phi^*(1)=0} = \frac{\hat{\phi}^*(1)}{se(\hat{\phi}^*(1))}$
  For critical values, see corresponding DF test.
- **Normalized bias statistic**: $N_{\phi^*(1)=0} = \frac{T*\hat{\phi}^*(1)}{1-\sum_{j=1}^{p-1}\hat{\varphi}_j}$
  For critical values, see corresponding DF test.

Both tests are left one-sided[20], thus:
. H_0 is rejected if $t < t^*_\alpha$ or $N < N_\alpha*$
. H_0 is not rejected if $t > t^*_\alpha$ or $N > N^*_\alpha$

- **Joint test (F)**
  Testing procedure is the same as in DF (likelihood ratio test with restricted and unrestricted models). We distinguish between 3 cases:
  - **Case 2**: $H_0 : \phi^*(1) = 0,\ \alpha = 0$. Thus, Restricted model is $\Delta X_t = \sum_{j=1}^{p-1} \varphi_j \Delta X_{t-j} + \epsilon_t$
    For critical values, see corresponding DF test, right one-sided.
  - **Case 4a**: $H_0 : \phi^*(1) = 0,\ \alpha = 0,\ \beta = 0$. Restricted model is $\Delta X_t = \sum_{j=1}^{p-1} \varphi_j \Delta X_{t-j} + \epsilon_t$
    For critical values, see corresponding DF test, right one-sided.
  - **Case 4b**: $H_0 : \phi^*(1) = 0,\ \beta = 0$. Restricted model is $\Delta X_t = \sum_{j=1}^{p-1} \varphi_j \Delta X_{t-j} + \epsilon_t$
    For critical values, see corresponding DF test, right one-sided.
- **Significance test (Student)**
  It's the same as in DF.
- **Testing φ_j**
  The coefficients for ΔX_t can be tested jointly or singularly using usual Student and Fisher statistics and critical levels. It's useful to choose the right number of AR lags (p_{max}).

3.5 Complete testing procedure

Except for the step wich involves 3.6, where we choose p_{max}, the steps are the same as in DF. Therefore, see subsection 2.5.

[19] Under white noise errors provided the number of lags is selected appropriately
[20] Here α stands for the confidence level and not for the constant term

3.6 Choosing p_{max}

ADF test are biased if the number of lags choosen is too small (i.e. there's still correlation in the residuals), while looses power if p_{max} is too large. Monte Carlo simulations indicate that the is preferable to exceed with lags than underestimate p_{max} . There are a few ways to chose the right number of lags.

1. Using appropriate information critiera (e.g. MAIC)
2. Use a rule of thumb
3. Test recursively the last two lags (t-test and F-test) of a regression and if they result equal to zero reduce number of lags.
4. Other ways (not treated)

4 Conclusion

This short paper doesn't treat the argument in a complete way. It's aimed only to resume the DF and ADF tests and give an overview on the testing procedure for unit roots with DF/ADF. Corrections and comments are welcome (igor.francetic@unil.ch).